(Knowledge for Development)

KIBABII UNIVERSITY COLLEGE

A CONSTITUENT COLLEGE OFMASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS

2014/2015 ACADEMIC YEAR

FOURTH YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

COURSE CODE: **MAT 424**

COURSE TITLE: **ODE III**

DATE: 29/4/15 **TIME:** **11.30AM -1.30PM**

INSTRUCTIONS TO CANDIDATES

Answer Question One in and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30MKS) (COMPULSORY)

a. Find the derivative of the function

$$f(x) = \begin{pmatrix} x_1 - x_2^2 \\ -x_2 + x_1 x_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

and evaluate it at the point $x_0 = (1, -1)^T$ (4 marks)

b. Solve the initial value problem

$$\dot{x} = a \quad , \quad x(0) = x_0$$

by method of successive approximations. (5 marks)

c. Prove that the function

$$V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4, \quad (y_1, y_2 \in \mathbb{R}$$

Is a strict Liapunov function of the system

$$\dot{x_1} = 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1 x_2^2$$
$$\dot{x_2} = x_2 - 2x_1 x_2 + x_1^2 x_2 - x_2^3$$

At the fixed point $(1,0)$ (6 marks)

d. Investigate the stability at the origin for the system (7 marks)

$$\dot{x_1} = 2x_2(x_3 - 1)$$
$$\dot{x_2} = -x_1(x_3 - 1)$$
$$\dot{x_3} = -x_3^3$$

e. Show that the phase portrait of

$$\ddot{x} - (1 - 3x^2 - 2\dot{x})\dot{x} + x = 0$$

has a limit cycle (8 marks)

QUESTION TWO (20 MARKS)

a. Investigate the stability of the second order equation

$$\ddot{x} + \dot{x}^3 + x = 0$$

at the origin of its phase plane (5 marks)

b. For the following functions

$$f(x) = \begin{pmatrix} x_1 + x_1 x_2^2 \\ -x_2 + x_2^2 + x_1^2 \end{pmatrix}, \quad f(x) = \begin{pmatrix} x_1 + x_1 x_2^2 + x_1 x_3^2 \\ -x_1 + x_2^2 - x_2 x_3 + x_1 x_2 x_3 \\ x_2 + x_3 - x_1^2 \end{pmatrix}$$

i. Compute the derivatives of these functions (5 marks)

ii. Find the zeros of the above functions, that is, the points $x_c \in \mathbb{R}$ where $f(x_c) = 0$ and evaluate $D \quad (x)$ at these points (8 marks)

iii. For the first function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined in part (i) above, compute $D^2 f(x_c)(x, y)$ where $x_c = (0,1)$ is a zero of f. (2 marks)

QUESTION THREE (20 MARKS)

a. Consider a predator-prey system governed by the following equations

$$\dot{x} = -a \quad + b$$
$$\dot{y} = c \quad - d$$

i. Which variable, x or y denotes the prey and predator species. Justify your answer. (2 marks)

ii. Find the equilibrium points for the above system and give its ecological interpretation (6 marks)

b. Consider the following logistic predator-prey model

$$\dot{x_1} = -x_1 + 0.9x_1x_2$$
$$x_2 = 2x_2\left(1 - \frac{x_2}{2}\right) - 1.2x_1x_2$$

i. Does $x_1(t)$ denote the prey or predator population? What of $x_2(t)$? Justify your answer. (2 marks)

ii. Find all equilibrium points (5 marks)

iii. Suppose the prey population becomes extinct while the predator population is still positive. Describe the long term behavior of the predator population (2 marks)

iv. Suppose the predator population becomes extinct while the prey population is still positive. Describe the long term behavior of the prey population (2 marks)

v. Describe the long-term behavior of the system when the initial population are given by

$$x_1(0) = \frac{20}{27} \quad a \qquad x_2(0) = \frac{10}{9}$$

(1 mark)

QUESTION FOUR (20 MARKS)

a. State and prove Grownwall's inequality (7 marks)

b. State the Poincaré-Bendixson theorem (3 marks)

c. State and prove the Bendixson's criterion (6 marks)

d. Using the Bendixson's criterion, determine whether the unforced Duffing oscillator

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_1 - x_1^3 - \delta x_2$$

has closed orbits lying it. (4 marks)

QUESTION FIVE (20 MARKS)

a. Consider the homogeneous linear periodic system

$$x = A(t)x, \quad A(t+T) = t, \quad t \in \mathbb{R}, \quad T > 0$$

Where $A(t)$ is continuous $n \times n$ real or complex matrix function of t containing x_0

Remark: C is an $n \times n$ matrix with d $C \neq 0$, then there is a matris B such that $C = e^B$

Using the above information, state and prove the Floquet theorem. (10 marks)

b. Find the maximal interval of existence for the following initial value problem

$$\dot{x} = x^2, \quad x(0) = x_0$$ (5 marks)

c. Show that the system

$$\dot{x_1} = x_1^2$$
$$\dot{x_2} = 2x_2^2 - x_1x_2$$

is unstable at the origin by using the function

$V(x) = \frac{1}{3}x_1^3 + 4x_1^2x_2 + 2x_1x_2^2 + \frac{4}{3}x_2^3$ (5 marks)