(Knowledge for Development)

KIBABII UNIVERSITY COLLEGE

A CONSTITUENT COLLEGE OFMASINDE MULIRO UNIVERSITY OF SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS

2014/2015 ACADEMIC YEAR

SECOND YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

MATHEMATICS

COURSE CODE: **MAT 202**

COURSE TITLE: **LINEAR ALGEBRA II**

DATE: 27/4/15 **TIME:** **8AM -10 AM**

INSTRUCTIONS TO CANDIDATES

Answer Question One in and Any other TWO Questions

TIME: 2 Hours

Question 1 (30 marks)

a. Prove the Parallelogram Law $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$. (3 marks)
b. Let W be the subspace of $\mathbf{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal complement $W^\perp$ of W. (4 marks)
c. Consider the subspace U of $\mathbf{R}^4$ spanned by the vectors $v_1 = (1, 1, 1, 1)$, $v_2 = (1, 1, 2, 4)$ and $v_3 = (1, 2, -4, -3)$. Find
   i. An orthogonal basis of U
   ii. An orthonormal basis of U. (5 marks)
d. Suppose that P is an orthogonal matrix. Show that $\langle P\ \ , P\ \ \rangle = \langle u, v\rangle$ (4 marks)
e. Find the characteristic polynomial of the linear operator F: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $F(x, y) = (3x + 5y, 2x - 7y)$. (2 marks)
f. State and prove the Cauchy – Schwarz Inequality. (6 marks)
g. Let S be a subset of an inner product space V. Show that the orthogonal complement of S, $S^\perp$ is a linear subspace of V. (4 marks)
h. Distinguish between a linear functional over a vector space V and the dual space of V. (2 marks)

Question Two (20 marks)

a. Find bases for the eigenspaces of of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \qquad \text{(8 marks)}$$

b. Define the geometric multiplicity and the algebraic multiplicity of an eigenvalue. State the value of each for the eigenvalues obtained in part (b) of this question. (4 marks)
c. Prove that λ^2 is an eigenvalue of A^2, if λ is an eigenvalue of the matrix A. (2 marks)
d. Prove that a square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A. (6 marks)

Question Three (20 marks)

a. State what, in Linear Algebra, is meant by a square matrix A being diagonalizable. (1 mark)
b. State how diagonalizability is related to the algebraic multiplicity and the geometric multiplicity of an eigenvalue of a given matrix. (4 marks)
c. Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$ (8 marks)

d. If A is an n x n matrix and P is an invertible matrix, prove that $(P^{-1}AP)^2 = P^{-1}A^2P$. Hence or otherwise show that $A^k = P\, D^k P^{-1}$. (7 marks)

Question Four (20 marks)

a. State and prove the triangle Inequality for inner product spaces. (4 marks)
b. Let W be a subspace of an inner product space V. Prove that $W^{\perp}$ is a subspace of V. (5 marks)
c. Consider the following polynomials in **P**(t) with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)d$. Let $g(t) = 3t - 2$ and $h(t) = t^2 - 2t - 3$. Determine
   i. $\langle g, h\rangle$
   ii. $||h||$
   iii. Normalize h (3 marks)
d. What is an invariant subspace? Suppose T: V→ V is linear. Show that each of the following in invariant under T.
   i. {0}
   ii. V
   iii. Kernel of T
   iv. Image of T (8 marks)

Question Five (20 marks)

a. Suppose matrix B is similar to matrix A, say $B = P^{-1}AP$. Prove that B^n is also similar to A^n. (7 marks)
b. Let F: $\mathbf{R}^2 \to \mathbf{R}^2$ be the linear operator defined by F(x, y) = (2x + 3y, 4x – 5y).

   Find the matrix representation of F relative to the basis

   $S = \{u_1, u_2\} = \{(1, 2), (2, 5)\}$. (6 marks)
   Find the matrix representation of F with respect to the usual standard basis. (2 marks)
c. Consider the following bases of $\mathbf{R}^2$; $E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ and $S = \{u_1, u_2\} = \{(1, 3), (1, 4)\}$. Find the change-of-basis matrix P from the usual basis E to S. (5 marks)