(Knowledge for Development)

KIBABII UNIVERSITY COLLEGE

A CONSTITUENT COLLEGE OFMASINDE MULIRO UNIVERSITY OF

SCIENCE AND TECHNOLOGY

UNIVERSITY EXAMINATIONS

2014/2015 ACADEMIC YEAR

THIRD YEAR SECOND SEMESTER

MAIN EXAMINATION

FOR THE DEGREE OF BACHELOR OF SCIENCE

MATHEMATICS

COURSE CODE: MAT 302

COURSE TITLE: REAL ANALYSIS III

DATE: 27/4/15 **TIME:** 8AM -10 AM

INSTRUCTIONS TO CANDIDATES

Answer Question One in and Any other TWO Questions

TIME: 2 Hours

QUESTION ONE (30 MARKS)

a) Let $\sum_{n\in IN} x_n$ be a series of elements of IK. Show that this series converges if and only if for each real number $\pounds > 0$ there is $N(\pounds) \in \mathbb{N}$ such that $\left|\sum_{k=m}^{n} x_k\right| < \pounds$ for all $n \geq m \geq N(\pounds)$. (4 mks)

b) Define the term absolute convergence of a series. Show that if $\sum_{n\in IN} z_n$ is absolutely convergent then $\sum_{n\in IN} z_n$ is convergent but the converse need not be true. (5mks)

c) Let $(X, \circ)$ be a compact metric space and $f:(X, \circ) \to (IR, d)$ be continuous. Show that f is bounded. If $m = \inf\{f(x): x \in X\}$ and $M = \sup\{f(x): x \in X\}$ show that there are points $x, x' \in X$ where $f(x) = m$ and $f(x') = M$. (4mks)

d) Show that all the monotonic functions on bounded intervals are functions of bounded variation. (3mks)

e) Define the term total variation. Let f be of bounded variation on the closed interval $[a,b]$ and $c \in (a,b)$ i.e. $a < c < b$. Show that $f \in BV[a,c]$, $f \in BV[c,b]$ and $V_f[a,b] = V_f[a,c] + V_f[c,b]$. (5mks)

f) Define what is meant by a function f being Riemann-Stieltjesintegrable. (5 marks)

g) Suppose $f:[a,b] \to IR$ is given and $\sim$ is increasing on $[a,b]$. If both f and $\sim$ have a right discontinuity (or left discontinuity) at some point c of $[a,b]$, show that $f \in \Re(\sim)$. (4 mks)

QUESTION TWO (20 MARKS)

a) Let $(x_n), (y_n)$ be convergent sequences of real numbers with limits x, y respectively and let $x_n \leq y_n \ \forall\ n \in \mathbb{N}$. Show that $x \leq y$. (4 marks)

b) Let $(x_n), (y_n), (z_n)$ be sequences of real numbers such that $x_n \leq z_n \leq y_n \ \forall\ n \geq N$ (N is a fixed integer). Let $(x_n), (y_n)$ both converge to the same limit say L, show that $z_n \to L$ as $n \to \infty$. (3 marks)

c) Show that $\lim_{n\to\infty} \sqrt[n]{n}$ exists and is 1. (4 marks)

d) Let f be of bounded variation on the closed interval $[a,b]$ and $x \in (a,b]$ i.e. $a < x \leq b$, then $f \in BV[a,x]$. Define a new function v on $[a,b]$ by

$$v(x) = \begin{cases} V_f[a,x] & \text{if } x \in (a,x] \\ 0 & \text{if } x = a \end{cases}.$$

i. Show that v is monotonic increasing on $[a,b]$.

ii. Let $D = v - f$. Show that D is monotonic increasing on $[a,b]$.

iii. Show that at each $x \in (a,b)$ the limits $f(x-), f(x+)$ exist. (9 marks)

QUESTION THREE (20 MARKS)

a) Show that the series $\sum_{n\in IN} \;\; ^{n}$ is convergent if $|\;\;|<1$ and divergent if $|\;\;|\geq 1$.

(4 marks)

b) Let $\sum_{n\in IN} a_n$ be a series of positive terms. Let $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ exist say L. Show that

i. If L<1 the series is convergent

ii. If L>1 the series is divergent. (6 marks)

c) Let $\sum_{n\in \mathrm{N}} a_n$ be a series of positive terms. Let $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ and $\lim_{n\to\infty} a_n^{\frac{1}{n}}$ both exist and equal to say L. Show that if L=1 then both the ratio and root tests fail the convergence test by using the series $\sum_{n\in IN}\frac{1}{n}$ and $\sum_{n\in IN}\frac{1}{n^2}$. (5 marks)

d) Let $f:[a,b]\to IR$ be bounded and $\sim$ be increasing on $[a,b]$. If $P_1, P_2 \in \mathbb{P}[a,b]$ and $P_1 \subseteq P_2$ (i.e. P_2 is finer than P_1), show that

i. $L(f,\sim,P_1)\leq L(f,\sim,P_2)$

ii. $U(f,\sim,P_1)\geq U(f,\sim,P_2)$ (5 marks)

QUESTION FOUR (20 MARKS)

a) Let $(X,\circ)$ be a metric space and $f:(X,\circ)\to(IR,d)$ be continuous. Let $a,b\in X$, $f(a)>0$ and $f(b)<0$. Show that there are neighbourhoods $N(a), N(b)$ of a, b respectively such that $f(x)>0\ \forall\ x\in N(a)$ and $f(x)<0\ \forall\ x\in N(b)$. (4 marks)

b) Let $f:[a,b]\to IR$ be continuous. Show that

i. If $f(a), f(b)$ have opposite signs i.e. $f(a)f(b)<0$, then there exists $c\in(a,b)$ i.e. $a<c<b$ such that $f(c)=0$.

ii. If $f(a)\neq f(b)$ and k is any number between $f(a)$ and $f(b)$, then there exists $c\in(a,b)$ i.e. $a<c<b$ such that $f(c)=k$.

(8 marks)

c) Let f be of bounded variation on the closed interval $[a,b]$. Show that f is bounded.

(4 marks)

d) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions defined on a set E of real numbers. Show that there exists a function f such that $f_n\to f$ uniformly on E if and only if the following (called the Cauchy condition) is satisfied; for every $£>0$ there exists an integer n_0 such that $m\geq n_0, n\geq n_0$ implies $|f_m(x)-f_n(x)|<£$ for every $x\in E$.

(4 marks)

QUESTION FIVE (20 MARKS)

a) Define the following terms
   i. Power series
   ii. Radius of convergence

(2 marks)

b) Let $f:(a,b)\to IR$ be monotonic. Then show that at each point $c\in(a,b)$ i.e. $a<c<b$, the one-sided limits $f(c-), f(c+)$ exist. Suppose f is increasing show that $f(c-)\leq f(c)\leq f(c+)$. If $x,y\in(a,b)$ and $x<y$, show that $f(x+)\leq f(y-)$. Moreover show that all points of discontinuity of f are of simple kind and the set of all points of discontinuity is at most countable.

(10 marks)

c) Let $f:[a,b]\to IR$ be bounded and $\sim$ be a monotonic increasing function on $[a,b]$. Show that $f\in\Re(\sim)$ on $[a,b]$ if and only if for every real £ >0 there exists $P\in\mathbb{P}[a,b]$ such that $U(f,\sim,P)-L(f,\sim,P)<$ £ .

(5 marks)

d) Define the terms pointwise convergence and uniform convergence in relation to sequence of functions.

(2 marks)